

Dickson's lemma and weak Ramsey theory

Yasuhiko Omata Florian Pelupessy*
Mathematical Institute, Tohoku University

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Abstract

We explore the connections between Dickson's lemma and weak Ramsey theory.

1 Introduction

Dickson's lemma, originally used in algebra, in particular for showing Hilbert's basis theorem, is nowadays commonly used in termination proofs in computer science [2]. The weak Paris–Harrington principle for pairs was originally used as an easy intermediate version in showing lower bounds for the Paris–Harrington principle for pairs [1]. We provide simple constructions which show that the weak Paris–Harrington principle and miniaturized Dickson's lemma are equivalent over RCA_0 . Additionally our construction provides an explicit formula for weak Ramsey numbers and tight upper bounds for the weak Paris–Harrington principle derived from those for Dickson's lemma.

\mathbb{N} denotes the set of non-negative integers. For $a, R, c \in \mathbb{N}$, $[a, R]^2$ denotes the set $\{(n, m) : a \leq n < m \leq R\}$ and c is identified with the set $\{n \in \mathbb{N} : n < c\}$. Given a coloring $C : [a, R]^2 \rightarrow c$, we say that a set $H \subseteq \mathbb{N}$ is *C-homogeneous* if $C|_{[H]^2}$ is constant. Similarly, we say that a set $H = \{h_0 < h_1 < \dots\} \subseteq \mathbb{N}$ is *C-weakly homogeneous* if $C(h_i, h_{i+1}) = C(h_{i+1}, h_{i+2})$ holds for all $h_i, h_{i+1}, h_{i+2} \in H$. Weakly homogeneous sets are sometimes called *adjacent homogeneous* or *path homogeneous*.

Definition 1 (WPH_f). For every $f : \mathbb{N} \rightarrow \mathbb{N}$ nondecreasing, the *weak Paris–Harrington principle for f* is the statement that for every a and c there exists R such that for every coloring $C : [a, R]^2 \rightarrow c$ there exists a C -weakly homogeneous set H with $|H| > f(\min H)$.

For c -tuples $\bar{m} = (m_0, \dots, m_{c-1}), \bar{n} = (n_0, \dots, n_{c-1}) \in \mathbb{N}^c$, define $\bar{m} \leq \bar{n}$ if and only if $\forall k < c$ ($m_k \leq n_k$), and $|\bar{m}|_\infty = \max_{k < c} \{m_k\}$.

Definition 2 (MDL_f). For every $f : \mathbb{N} \rightarrow \mathbb{N}$ nondecreasing, the *miniaturized Dickson's lemma for f* is the statement that for every a and c there exists D such that for every

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sequence $\bar{m}_0, \dots, \bar{m}_D \in \mathbb{N}^c$ with $|\bar{m}_i|_\infty < f(a+i)$ there exists $i < j \leq D$ such that $\bar{m}_i \leq \bar{m}_j$.

Our original intent was to provide an easy proof of equivalence of DL and $\forall f\text{WPH}_f$ (Corollary 15) and equivalence of WPH_{id} and MDL_{id} (Corollary 13). With some work, this can already be shown using proofs of equivalences of

- PH_{id} and $1\text{-Con}(\text{IS}_1)$ ([4]),
- $\forall f\text{PH}_f$ and $\text{WO}(\omega^\omega)$ ([5]),
- DL and $\text{WO}(\omega^\omega)$ ([7]).

Our method additionally gives an improved result stated in Corollaries 7, 9 and the explicit expression in Theorem 11 for the weak Ramsey numbers. These results may be of particular interest in computer science.

The proof of the Lemma 5 (i) was originally presented by the second author.

2 Constructions

Definition 3 (bad coloring). Given $a, c, R \in \mathbb{N}$ and $f: \mathbb{N} \rightarrow \mathbb{N}$ nondecreasing, a coloring $C: [a, R]^2 \rightarrow c$ is *f-bad* if every C -weakly homogeneous set $H \subseteq [a, R]$ has size $\leq f(\min H)$.

Definition 4 (bad sequence). Given $a, c, D \in \mathbb{N}$ and $f: \mathbb{N} \rightarrow \mathbb{N}$ nondecreasing, a sequence $\bar{m}_0, \dots, \bar{m}_D \in \mathbb{N}^c$ is *bad* if for all $i < j \leq D$, $\bar{m}_i \not\leq \bar{m}_j$, we call it (a, f) -*bad* if additionally $|\bar{m}_i|_\infty < f(a+i)$.

WPH_f states that for every a and c there exists R such that there is no f -bad coloring $C: [a, R]^2 \rightarrow c$. MDL_f states that for every a and c there exists D such that there is no (a, f) -bad sequence $\bar{m}_0, \dots, \bar{m}_D \in \mathbb{N}^c$.

Lemma 5. RCA_0 proves:

- Existence of a f -bad coloring $C: [a, R]^2 \rightarrow c$ implies existence of a (a, f) -bad sequence $\bar{m}_0, \dots, \bar{m}_{R-a} \in \mathbb{N}^c$;
- Existence of a (a, f) -bad sequence $\bar{m}_0, \dots, \bar{m}_D \in \mathbb{N}^c$ implies existence of a f -bad coloring $C: [a, a+D]^2 \rightarrow c$.

The same hold for bad colorings $C: [a, \infty]^2 \rightarrow c$ and infinite (a, f) -bad sequences.

Proof of (i). Let $C: [a, R]^2 \rightarrow c$ be a given f -bad coloring. The idea of construction is as follows: take the (a, f) -bad sequence such that if $C(a+j, a+i) = k$ then $(\bar{m}_j)_k > (\bar{m}_i)_k$, and such that the sup-norm of the elements is the maximum allowed.

Define $h: [0, R-a] \times c \rightarrow \mathbb{N}$ primitive-recursively as follows:

$$h(i, k) = \begin{cases} \min S_{i,k} & S_{i,k} \neq \emptyset \\ f(a+i) - 1 & S_{i,k} = \emptyset \end{cases}$$

where $S_{i,k} = \{h(j,k) - 1 : j < i \text{ \& } C(a+j, a+i) = k\}$.

We show that $h(i,k) \geq 0$ whenever this value defined. For a contradiction, assume that $h(j^{(0)}, k) = -1$ for some $j^{(0)}, k$.

By the definition of h , there exists $l \in \mathbb{N}$ such that

$$\begin{aligned} \exists j^{(1)} < j^{(0)} \text{ s.t. } h(j^{(1)}, k) = 0 \text{ \& } C(a+j^{(1)}, a+j^{(0)}) = k, \\ \exists j^{(2)} < j^{(1)} \text{ s.t. } h(j^{(2)}, k) = 1 \text{ \& } C(a+j^{(2)}, a+j^{(1)}) = k, \\ & \vdots \\ \exists j^{(l)} < j^{(l-1)} \text{ s.t. } h(j^{(l)}, k) = l-1 = f(a+j^{(l)}) - 1 \\ & \text{\& } C(a+j^{(l)}, a+j^{(l-1)}) = k. \end{aligned}$$

Hence $H = \{a+j^{(l)} < a+j^{(l-1)} < \dots < a+j^{(0)}\}$ is a C -weakly homogeneous set of size $l+1$. But $f(\min H) = f(j^{(l)} + a) = l \not\geq |H| = l+1$, this contradicts to the assumption that C is f -bad.

Define $\bar{m}_i = (h(i,0), \dots, h(i, c-i)) \in \mathbb{N}^c$ for each $i \leq R-a$ and $k < c$. Then the sequence $\bar{m}_0, \dots, \bar{m}_{R-a}$ is bad, because $i < j \leq R-a$ implies $(\bar{m}_i)_k = h(i,k) > h(j,k) = (\bar{m}_j)_k$ where $k = C(i+a, j+a)$. Moreover for each \bar{m}_i we have $|\bar{m}_i|_\infty < f(a+i)$ by the definition of h . This completes the proof.

Proof of (ii). Let $\bar{m}_0, \dots, \bar{m}_D$ be a given (a, f) -bad sequence. Since this is bad, for every $i < j \leq D$ there is a $k \in \mathbb{N}$ such that $(\bar{m}_i)_k > (\bar{m}_j)_k$. We choose the smallest such $k = k(i, j)$ for each $i < j \leq D$, and define a coloring $C: [a, a+D]^2 \rightarrow c$ by $C(a+i, a+j) = k(i, j)$. To show that C is a f -bad coloring, suppose $H = \{a+h_0 < a+h_1 < \dots\} \subseteq [a, a+D]$ is a C -weakly homogeneous set. Then $(\bar{m}_{h_0})_k > (\bar{m}_{h_1})_k > \dots$ for some $k < c$. Since these values are all non-negative, maximum possible size of H is $(\bar{m}_{h_0})_k + 1 \leq |\bar{m}_{h_0}|_\infty + 1 \leq f(a+h_0) = f(\min H)$. This completes the proof. \square

3 Complexities

Definition 6 (R_c^f and D_c^f). For c and f nondecreasing, take

$$\begin{aligned} R_c^f(a) &= \text{the smallest } R \text{ s.t. there is no } f\text{-bad coloring } C: [a, R]^2 \rightarrow c, \\ D_c^f(a) &= \text{the smallest } D \text{ s.t. there is no } (a, f)\text{-bad sequence } \bar{m}_0, \dots, \bar{m}_D \in \mathbb{N}^c. \end{aligned}$$

By Lemma 5, we immediately have:

Corollary 7. $R_c^f(a) = D_c^f(a) + a$ holds for every a, c and f nondecreasing.

Remark. This equation depends on the formulations of WPH_f and DL_f . If one formulates WPH_f as “ $\forall a, c \exists R \forall C: [0, R]^2 \rightarrow c \exists H: C\text{-weakly homogeneous with } |H| > f(a + \min H)$ ”, one will get $R_c^f(a) = D_c^f(a)$.

Definition 8 (Fast growing hierarchy, [6]). Define functions $F_n: \mathbb{N} \rightarrow \mathbb{N}$ inductively as follows:

$$\begin{aligned} F_0(x) &= x + 1; \\ F_{n+1}(x) &= F_n^{(x)}(x). \end{aligned}$$

Then, \mathfrak{F}_n is the smallest which class contains constants, sum, projections, and F_n , and closed under the operations of composition (substitution) and limited primitive recursion.

Corollary 9. *Let $n \geq 1$. For all f nondecreasing with $f(x) \geq \max\{1, x\}$, if $f \in \mathfrak{F}_n$ then $R_c^f \in \mathfrak{F}_{n+c-1}$.*

Proof. By Corollary 7 and [2, Proposition V.2]. □

Definition 10 (r_c and wr_c). *Finite Ramsey's theorem for pairs*, denoted FRT, is the statement says that for every a, c there exists R such that for every coloring $C: [R]^2 \rightarrow c$ there exists a C -homogeneous set H with $|H| > a$, where $[R]^2 = [0, R]^2$. *Weak finite Ramsey's theorem for pairs*, denoted WFRT, is FRT but C -weakly homogeneous set instead of C -homogeneous set. Define

$$\begin{aligned} r_c(a) &= \text{the smallest } R \text{ s.t. for every } C: [R]^2 \rightarrow c \\ &\quad \text{there exists a } C\text{-homogeneous set } H \text{ with } |H| = a + 1, \\ wr_c(a) &= \text{the smallest } R \text{ s.t. for every } C: [R]^2 \rightarrow c \\ &\quad \text{there exists a } C\text{-weakly homogeneous set } H \text{ with } |H| = a + 1. \end{aligned}$$

Clearly $wr_c(a) \leq r_c(a)$. These are the smallest witnesses for FRT and WFRT respectively.

Theorem 11. $wr_c(a) = a^c$.

Proof. For each a , let f_a be the constant function $f_a(n) = a$. We have $wr_c(a) = R_c^{f_a}(0) = D_c^{f_a}(0)$ by Corollary 7. Moreover $D_c^{f_a}(0) = a^c$, since $D_c^{f_a}(0) \leq a^c$ by finite pigeonhole principle, and $D_c^{f_a}(0) > a^c - 1$ by existence of the bad sequence enumerating c -tuples in $\{0, \dots, a-1\}^c$ in decreasing lexicographical order. □

Corollary 12. $r_c(a) \geq a^c$.

Remark that this lower bound for the Ramsey number is not good compared to known results; see [3]. Our advantage is that the proof is very easy.

4 Reverse Mathematics

By Lemma 5, we also have:

Corollary 13. $\text{RCA}_0 \vdash \forall f: \text{nondecreasing} \left(\text{MDL}_f \leftrightarrow \text{WPH}_f \right)$.

Definition 14 (DL and RPH). *Dickson's Lemma* is the statement that for every c and an infinite sequence $\bar{m}_0, \bar{m}_1, \dots \in \mathbb{N}^c$ there exists $i < j$ such that $\bar{m}_i \leq \bar{m}_j$. *The Relativized weak Paris–Harrington principle* is the statement that for every $f: \mathbb{N} \rightarrow \mathbb{N}$ WPH_f holds.

Corollary 15.

$$\text{RCA}_0 \vdash \text{DL} \leftrightarrow \text{RPH}.$$

Proof. For $(\text{DL} \rightarrow \text{RPH})$, we assume $\neg \text{RPH}$. Then there exists $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\neg \text{WPH}_f$ holds, i.e. there exists $a, c \in \mathbb{N}$ such that for all R there exists f -bad coloring $C: [a, R]^2 \rightarrow c$. Assume without loss of generality that f is nondecreasing. By Lemma 5 (i), it follows that there is arbitrarily long (finite) (a, f) -bad sequence $\bar{m}_0, \bar{m}_1, \dots \in \mathbb{N}^c$.

We show there is an infinite bad sequence. Let $T \in \mathbb{N}^{<\mathbb{N}}$ be the tree consisting of (the codes of) (a, f) -bad sequences $\langle \bar{m}_0, \bar{m}_1, \dots \rangle$. Then T is infinite and bounded, since our code of c -tuple \bar{m}_i is bounded exponentially in $f(a+i)$. By bounded König's lemma, T has an infinite path which shows $\neg \text{DL}$. Since bounded König's lemma is equivalent to weak König's lemma over RCA_0 [8, Lemma IV.1.4], we have shown $\text{WKL}_0 + \text{DL} \vdash \text{RPH}$. By Π_1^1 -conservativity of $\text{WKL}_0 + \text{DL}$ over $\text{RCA}_0 + \text{DL}$ [5, Theorem 20], we have $\text{RCA}_0 + \text{DL} \vdash \text{RPH}$.

For $(\text{RPH} \rightarrow \text{DL})$, we assume $\neg \text{DL}$. Then there exists infinite $(0, f)$ -bad sequence $\bar{m}_0, \bar{m}_1, \dots \in \mathbb{N}^c$ with $|\bar{m}_i|_\infty < f(i)$, where $f(i) = \max_{j \leq i} |\bar{m}_j|_\infty + 1$. By Lemma 5 (ii), this realizes the existence of f -bad coloring $C: [0, R]^2 \rightarrow c$ for every R , which is exactly $\neg \text{RPH}$. \square

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